

A note on the minimum skew rank of a graph

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Abstract

The minimum skew rank $mr^-(\mathbb{F}, G)$ of a graph G over a field \mathbb{F} is the smallest possible rank among all skew symmetric matrices over \mathbb{F} , whose (i, j) -entry (for $i \neq j$) is nonzero whenever ij is an edge in G and is zero otherwise. We give some new properties of the minimum skew rank of a graph, including a characterization of the graphs G with cut vertices over the infinite field \mathbb{F} such that $mr^-(\mathbb{F}, G) = 4$, determination of the minimum skew rank of k -paths over a field \mathbb{F} , and an extending of an existing result to show that $mr^-(\mathbb{F}, G) = 2\text{match}(G) = MR^-(\mathbb{F}, G)$ for a connected graph G with no even cycles and a field \mathbb{F} , where $\text{match}(G)$ is the matching number of G , and $MR^-(\mathbb{F}, G)$ is the largest possible rank among all skew symmetric matrices over \mathbb{F} .

Key words: Minimum skew rank, Skew-symmetric matrix, k -tree, k -path, Zero forcing number, Perfect matching

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1 Introduction

We consider only simple graphs. Let G be a graph with vertex set V_G and edge set E_G . Let \mathbb{F} be a field. We adopt the notation and terminology from [5] and [8].

An $n \times n$ matrix A over \mathbb{F} is skew-symmetric (respectively, symmetric) if $A^T = -A$ (respectively, $A^T = A$), where A^T denotes the transpose of A .

For an $n \times n$ symmetric or skew-symmetric matrix A , the graph of A , denoted $G(A)$, is the graph with vertex set $\{v_1, v_2, \dots, v_n\}$ and edge set $\{v_i v_j : a_{ij} \neq 0, 1 \leq i < j \leq n\}$.

The classic minimum rank problem involving symmetric matrices has been studied extensively, see, e.g., [4].

The minimum skew rank problem involves skew symmetric matrices and its study began recently in [5]. If the characteristic of \mathbb{F} is 2, then a skew-symmetric matrix over \mathbb{F} is also symmetric. Thus it is assumed throughout this paper that the characteristic of \mathbb{F} is not 2.

For a field \mathbb{F} and a graph G , let $S^-(\mathbb{F}, G) = \{A \in \mathbb{F}^{n \times n} : A^T = -A, G(A) = G\}$ be the set of skew-symmetric matrices over \mathbb{F} described by G . The minimum skew rank of G over \mathbb{F} is defined as

$$mr^-(\mathbb{F}, G) = \min\{\text{rank}(A) : A \in S^-(\mathbb{F}, G)\}.$$

The corresponding maximum skew nullity of G is defined as

$$M^-(\mathbb{F}, G) = \max\{\text{nullity}(A) : A \in S^-(\mathbb{F}, G)\}.$$

Obviously, $mr^-(\mathbb{F}, G) + M^-(\mathbb{F}, G) = |V_G|$.

Let K_n be the complete graph with n vertices, and K_{n_1, n_2, \dots, n_t} the complete t -partite graph with n_i vertices in the i th partite sets for $i = 1, 2, \dots, t$.

Note that the rank of a skew-symmetric matrix over \mathbb{F} is always even. Thus $mr^-(\mathbb{F}, G)$ is even for any field \mathbb{F} and any graph G . As observed in [5], $mr^-(\mathbb{F}, G) = 0$ if and only if G is an empty graph, and if \mathbb{F} is infinite and G is a connected graph with at least two vertices, then $mr^-(\mathbb{F}, G) = 2$ if and only if G is a complete multipartite graph K_{n_1, n_2, \dots, n_t} for some $t \geq 2$, $n_i \geq 1$ for $i = 1, \dots, t$. The authors [5] posed an open question (Question 5.2) to characterize the graphs G such that $mr^-(\mathbb{F}, G) = 4$. We characterize the graphs G with cut vertices over the infinite field \mathbb{F} such that $mr^-(\mathbb{F}, G) = 4$.

The class of k -trees is defined recursively as follows [6]: (i) The complete graph K_{k+1} is a k -tree; (ii) A k -tree G with $n+1$ vertices ($n \geq k+1$) can be constructed from a k -tree H on n vertices by adding a vertex adjacent to all vertices of a k -clique of H . A k -path is a k -tree which is either K_{k+1} or has exactly two vertices of degree k . We determine the minimum skew rank of k -paths over a field \mathbb{F} . The k -th power G^k of a graph G is the graph whose vertex set is V_G , two distinct vertices being adjacent in G^k if and only if their distance in G is at most k . Let $P_n = v_1 v_2 \dots v_n$ be the path on n vertices. If $k \leq n-1$, then P_n^k is a k -path (see below). As a corollary, we obtain the

minimum skew rank of the k -th power of a path over the real field \mathbb{R} , which was already given in [2].

The maximum skew rank $MR^-(\mathbb{F}, G)$ of a graph G over a field \mathbb{F} is defined as

$$MR^-(\mathbb{F}, G) = \max\{\text{rank}(A) : A \in S^-(\mathbb{F}, G)\}.$$

Let $\text{match}(G)$ be the matching number of G . It was shown in [5] that $mr^-(\mathbb{F}, G) = 2\text{match}(G) = MR^-(\mathbb{F}, G)$ for a tree (a connected graph with no cycles) G and a field \mathbb{F} . We extend this by showing that the above conclusion holds also for a connected graph G with no even cycles.

2 Preliminaries

Let G be a graph. For $v \in V_G$, $G - v$ denotes the graph obtained from G by deleting vertex v (and all edges incident with v). For $X \subseteq V_G$, $G[X]$ denotes the subgraph of G induced by vertices in X .

We give some lemmas that we will use in our proof.

Lemma 2.1 [5] *Let G be a connected graph with at least two vertices and let \mathbb{F} be an infinite field. Then $mr^-(\mathbb{F}, G) = 2$ if and only if G is a complete multipartite graph.*

For a field \mathbb{F} and a graph G with $v \in V_G$, let $r_v^-(\mathbb{F}, G) = mr^-(\mathbb{F}, G) - mr^-(\mathbb{F}, G - v)$.

The union of graphs G_i , $i = 1, 2, \dots, h$, denoted by $\cup_{i=1}^h G_i$, is the graph with vertex set $\cup_{i=1}^h V_{G_i}$ and edge set $\cup_{i=1}^h E_{G_i}$.

Lemma 2.2 [5, 3] *Let G be a graph with cut vertex v and \mathbb{F} a field, where $G = \cup_{i=1}^h G_i$ and $\cap_{i=1}^h V_{G_i} = \{v\}$. Then $mr^-(\mathbb{F}, G) = \sum_{i=1}^h mr^-(\mathbb{F}, G_i - v) + \min\{\sum_{i=1}^h r_v^-(\mathbb{F}, G_i), 2\}$.*

Lemma 2.3 [5] *Let G be a graph and let \mathbb{F} be an infinite field. If $G = G_1 \cup G_2$, then $mr^-(\mathbb{F}, G) \leq mr^-(\mathbb{F}, G_1) + mr^-(\mathbb{F}, G_2)$.*

Let G be a graph. A subset $Z \subset V_G$ defines an initial coloring by coloring all vertices in Z black and all the vertices outside Z white. The color change rule says: If a black vertex u has exactly one white neighbor v , then change the color of v to black. In this case we write $u \rightarrow v$. The derived set of an

initial coloring Z is the set of vertices colored black until no more changes are possible. A zero forcing set is a subset $Z \subset V_G$ such that the derived set of Z is V_G . The zero forcing number of G , denoted by $Z(G)$, is the minimum size of a zero forcing set of G .

Lemma 2.4 [5] *Let G be a graph and \mathbb{F} a field. Then $M^-(\mathbb{F}, G) \leq Z(G)$.*

Lemma 2.5 [5] *Let G be a graph and \mathbb{F} a field. Then $MR^-(\mathbb{F}, G) = 2\text{match}(G)$.*

Lemma 2.6 [5] *Let G be a graph and \mathbb{F} a field. If H is an induced subgraph of G , $mr^-(\mathbb{F}, H) \leq mr^-(\mathbb{F}, G)$.*

Lemma 2.7 [5] *Let G be a graph with a unique perfect matching and \mathbb{F} a field. Then $mr^-(\mathbb{F}, G) = |V_G|$.*

3 Results

we first give a partial result to the question in [5] to characterize graphs with $mr^-(\mathbb{F}, G) = 4$. We consider graphs with cut vertices.

Theorem 3.1 *Let G be a graph with cut vertex v and \mathbb{F} an infinite field. Then $mr^-(\mathbb{F}, G) = 4$ if and only if one of the following conditions holds:*

- (i) $G = G_1 \cup G_2$ and $V_{G_1} \cap V_{G_2} = \{v\}$, where G_1, G_2 are complete multipartite graphs such that $G_1 - v, G_2 - v$ are nonempty, and
- (ii) $G - v$ consists of a nonempty complete multipartite component and isolated vertices.

Proof. Suppose first that (i) holds. Note that $G_i - v$ is still a complete multipartite graph for $i = 1, 2$. By Lemma 2.1, $mr^-(\mathbb{F}, G_1) = mr^-(\mathbb{F}, G_2) = mr^-(\mathbb{F}, G_1 - v) = mr^-(\mathbb{F}, G_2 - v) = 2$. Then $r_v^-(\mathbb{F}, G_1) + r_v^-(\mathbb{F}, G_2) = 0$. Thus by Lemma 2.2, $mr^-(\mathbb{F}, G) = mr^-(\mathbb{F}, G_1 - v) + mr^-(\mathbb{F}, G_2 - v) + \min\{0, 2\} = 4$.

Now suppose that (ii) holds. Let W be the unique complete multipartite component, and a the number of isolated vertices in $G - v$. By Lemma 2.1, $mr^-(\mathbb{F}, W) = 2$. Note that $r_v^-(\mathbb{F}, K_2) = 2$. Then by Lemma 2.2, $mr^-(\mathbb{F}, G) = mr^-(\mathbb{F}, W) + a \cdot mr^-(\mathbb{F}, K_1) + 2 = 4$.

Conversely, suppose that $mr^-(\mathbb{F}, G) = 4$. Let p be the number of nonempty complete multipartite components, and q the number of isolated vertices in $G - v$. Let m be the number of the remaining components.

Note that the minimum skew rank of a graph that is neither a complete multipartite graph nor an empty graph is larger than 4.

Case 1. $q = 0$. By Lemma 2.2, $4 = mr^-(\mathbb{F}, G) \geq 2p + 4m$. If $m = 1$, then $p = 0$, a contradiction to the fact that v is a cut vertex of G . Thus $m = 0$, implying that $p = 2$. Let W_1, W_2 be the vertex sets of the two complete multipartite components of $G - v$ and let G_1, G_2 be the subgraph induced by $\{v\} \cup W_1, \{v\} \cup W_2$. By Lemma 2.1, $mr^-(\mathbb{F}, G_1 - v) = mr^-(\mathbb{F}, G_2 - v) = 2$. By Lemma 2.2, $4 = mr^-(\mathbb{F}, G) = mr^-(\mathbb{F}, G_1 - v) + mr^-(\mathbb{F}, G_2 - v) + \min\{r_v^-(\mathbb{F}, G_1) + r_v^-(\mathbb{F}, G_2), 2\} = 2 + 2 + \min\{r_v^-(\mathbb{F}, G_1) + r_v^-(\mathbb{F}, G_2), 2\}$. Then $r_v^-(\mathbb{F}, G_1) = r_v^-(\mathbb{F}, G_2) = 0$. Thus $mr^-(\mathbb{F}, G_1) = mr^-(\mathbb{F}, G_2) = 2$. By Lemma 2.1, G_1 and G_2 are complete multipartite graphs, and then (i) follows.

Case 2. $q \neq 0$. Note that $r_v^-(\mathbb{F}, K_2) = 2$. By Lemma 2.2, $4 = mr^-(\mathbb{F}, G) \geq 2p + 4m + 2$. Then $m = 0$ and $p = 1$, and thus (ii) follows. \square

Now we consider the minimum skew rank of k -paths. Note that a k -path with at least $k + 2$ vertices has at least two vertices of degree k and any two vertices of degree k are not adjacent. The following lemma follows directly from the definition of k -path.

Lemma 3.1 *Let G be a k -path with at least $k + 2$ vertices, and v a vertex of G with degree k . Then $G - v$ is also a k -path.*

Let G be a k -path with $n \geq k + 2$ vertices. By Lemma 3.1, the vertices of G may be labeled as follows: Choose a vertex of degree k , labeled as v_n , and label its unique neighbor of degree $k + 1$ in G with v_{n-1} . Then v_{n-1} is a vertex of degree k in the k -path $G - v_n$. Repeating the process above, we may label $n - k + 1$ vertices of G as $v_n, v_{n-1}, \dots, v_{k+2}$. Obviously, $G - v_n - v_{n-1} - \dots - v_{k+2} = K_{k+1}$ and it contains a vertex of degree k in G , which is labeled as v_1 , and the remaining vertices are labeled as v_2, v_3, \dots, v_{k+1} such that v_2 is the unique neighbor of v_1 with degree $k + 1$ in G . Note that in our labelling, v_i is not adjacent to $v_{j+1}, v_{j+2}, \dots, v_n$ if v_i is not adjacent to v_j for $j \geq \max\{i + 1, k + 2\}$. Recall that a k -tree is a chordal graph. The above labeling is actually the “perfect elimination” labeling inherent to chordal graphs [7].

Theorem 3.2 *Let G be a k -path on n vertices and \mathbb{F} an infinite field. Then*

$$mr^-(\mathbb{F}, G) = \begin{cases} n - k & \text{if } n - k \text{ is even,} \\ n - k + 1 & \text{if } n - k \text{ is odd.} \end{cases}$$

Proof. Let $Z = \{v_1, v_2, \dots, v_k\}$. Color all vertices in Z black and all the vertices outside Z white. We will show that Z is a zero forcing set of G . Since all neighbors of v_1 different from v_{k+1} are black, we have $v_1 \rightarrow v_{k+1}$. Note that v_2 is adjacent to v_{k+2} but not adjacent to $v_{k+3}, v_{k+4}, \dots, v_n$. Since all neighbors of v_2 different from v_{k+2} are black, we have $v_2 \rightarrow v_{k+2}$. Let $G_1 = G[\{v_1, v_2, \dots, v_{k+3}\}]$ and $G_2 = G[\{v_1, v_2, \dots, v_{k+4}\}]$. If each neighbor of v_{k+3} in G_1 is adjacent to v_{k+4} in G , then v_{k+4} is of degree $k+1$ in G_2 , a contradiction. Thus there is a neighbor, say w , of v_{k+3} in G_1 such that $wv_{k+4} \notin E_G$, and then $wv_i \notin E_G$ for $i \geq k+5$, implying that $w \rightarrow v_{k+3}$. Repeating the process above, we may finally color all vertices of G black. Thus Z is a zero forcing set of G . By Lemma 2.4, $M^-(\mathbb{F}, G) \leq Z(G) \leq k$, and then $mr^-(\mathbb{F}, G) = n - M^-(\mathbb{F}, G) \geq n - k$. Note that the rank of a skew-symmetric matrix is even. It follows that

$$mr^-(\mathbb{F}, G) \geq \begin{cases} n - k & \text{if } n - k \text{ is even,} \\ n - k + 1 & \text{if } n - k \text{ is odd.} \end{cases}$$

To prove the result, we need only to show

$$mr^-(\mathbb{F}, G) \leq \begin{cases} n - k & \text{if } n - k \text{ is even,} \\ n - k + 1 & \text{if } n - k \text{ is odd.} \end{cases} \quad (1)$$

We prove this by induction on n . If $n = k + 1$, then $G = K_{k+1}$, which is a complete multipartite graph, and thus by Lemma 2.1, $mr^-(\mathbb{F}, G) = 2 = n - k + 1$. If $n = k + 2$, then $G = K_{k+2} - e$ is also a complete multipartite graph, where $e \in E_{K_{k+2}}$, and thus by Lemma 2.1, $mr^-(\mathbb{F}, G) = 2 = n - k$. Thus (1) is true for $n = k + 1, k + 2$. Suppose that $n \geq k + 3$ and for a k -path H on m vertices with $k + 1 \leq m \leq n - 1$, we have

$$mr^-(\mathbb{F}, H) \leq \begin{cases} m - k & \text{if } m - k \text{ is even,} \\ m - k + 1 & \text{if } m - k \text{ is odd.} \end{cases}$$

Let G be a k -path on n vertices. Let

$$G_1 = G[\{v_1, v_2, \dots, v_{k+2}\}] \text{ and } G_2 = G[\{v_3, v_4, \dots, v_n\}].$$

Then G_1 is a k -path on $k + 2$ vertices, and G_2 is a k -path on $n - 2$ vertices. Obviously, $mr^-(\mathbb{F}, G_1) = 2$, and by the induction hypothesis,

$$mr^-(\mathbb{F}, G_2) \leq \begin{cases} n - k - 2 & \text{if } n - k - 2 \text{ is even,} \\ n - k - 1 & \text{if } n - k - 2 \text{ is odd,} \end{cases}$$

i.e.,

$$mr^-(\mathbb{F}, G_2) \leq \begin{cases} n - k - 2 & \text{if } n - k \text{ is even,} \\ n - k - 1 & \text{if } n - k \text{ is odd.} \end{cases}$$

Note that $G = G_1 \cup G_2$. By Lemma 2.3,

$$\begin{aligned} mr^-(\mathbb{F}, G) &\leq mr^-(\mathbb{F}, G_1) + mr^-(\mathbb{F}, G_2) \\ &\leq 2 + \begin{cases} n - k - 2 & \text{if } n - k \text{ is even} \\ n - k - 1 & \text{if } n - k \text{ is odd} \end{cases} \\ &= \begin{cases} n - k & \text{if } n - k \text{ is even,} \\ n - k + 1 & \text{if } n - k \text{ is odd.} \end{cases} \end{aligned}$$

This proves (1). □

Obviously, P_n^k is a complete graph if $k \geq n$. Suppose that $k \leq n - 1$. Obviously, $P_n^k[\{v_1, v_2, \dots, v_{k+1}\}] = K_{k+1}$, and if $k \leq n - 2$, then for $j = 2, 3, \dots, n - k$, $P_n^k[\{v_j, v_{j+1}, \dots, v_{k+j-1}\}] = K_k$, and v_{k+j} is adjacent to $v_j, v_{j+1}, \dots, v_{k+j-1}$. Thus P_n^k is a k -path. Now by Lemma 2.1 and Theorem 3.2 we have the following result, which was proved in [2] when \mathbb{F} is the real field \mathbb{R} .

Corollary 3.1 *Let \mathbb{F} be an infinite field. Then*

$$mr^-(\mathbb{F}, P_n^k) = \begin{cases} n - k & \text{if } 1 \leq k \leq n - 1 \text{ and } n - k \text{ is even,} \\ n - k + 1 & \text{if } 1 \leq k \leq n - 1 \text{ and } n - k \text{ is odd,} \\ 2 & \text{if } k \geq n. \end{cases}$$

Finally, we gave an observation.

Theorem 3.3 *Let G be a connected graph with no even cycles and \mathbb{F} a field. Then $mr^-(\mathbb{F}, G) = 2\text{match}(G) = MR^-(\mathbb{F}, G)$.*

Proof. By Lemma 2.5, $mr^-(\mathbb{F}, G) \leq MR^-(\mathbb{F}, G) = 2\text{match}(G)$. Let M be a maximum matching of G and $\{v_1, \dots, v_k\}$ the vertices in M . Then M is a perfect matching of $H = G[\{v_1, \dots, v_k\}]$. This perfect matching is unique. Otherwise, the graph induced by the vertices of the symmetric difference of two (different) perfect matchings of H consists of even cycles,

which is impossible because G contains no even cycles. By Lemmas 2.6 and 2.7, $mr^-(\mathbb{F}, G) \geq mr^-(\mathbb{F}, H) = 2\text{match}(G)$. The result follows. \square

Note that a tree has no (even) cycles. By previous theorem we have the following result.

Corollary 3.2 [5] *Let G be a tree and \mathbb{F} a field. Then $mr^-(\mathbb{F}, G) = 2\text{match}(G) = MR^-(\mathbb{F}, G)$.*

Let G be a connected unicyclic graph with a unique cycle C . If C is odd, then by Theorem 3.3, $mr^-(\mathbb{F}, G) = 2\text{match}(G)$. Recall that it was shown in [1] that if C is odd, then $mr^-(\mathbb{R}, G) = 2\text{match}(G)$, and if C is even, then $mr^-(\mathbb{R}, G) = 2\text{match}(G)$ or $2\text{match}(G) - 2$.

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